1. *Let A, B, C be non-empty sets and let $f : A \to B; g : B \to C$ and $h : A \to C$ be functions with $h = g \circ f$ (composition). Prove or disprove the following:*

   (a) *If f, g are surjective then h is surjective.*

   (b) *If h is surjective then f, g are surjective.*

   (c) *If h is injective then f, g are injective.*

   Solution:

   (a) Given that f, g are surjective
   i.e., $f(A) = B, g(B) = C$. This implies that $h(A) = g(f(A)) = g(B) = C$
   Hence h is surjective

   (b) Consider $A = \{a, b\}, B = \{1, 2, 3\}, C = \{\alpha, \beta\}$ and f, g be defined as $f(a) = 1, f(b) = 2, g(1) = \alpha, g(2) = g(3) = \beta$. Then $h(A) = C$. i.e., h is surjective. but $f(A) = \{1, 2\} \neq B$. i.e., f is not surjective.

   (c) Consider A, B, C, f, g as in (b) $h(a) = \alpha, h(b) = \beta$ implies that h is injective but $g(2) = g(3)$. Hence g is not injective.

   $\square$

2. *Let $X = \{1, 2\}$ and let M be the set of all integer valued functions on X. Show that M is countable.*

   Solution: Let $X = \{1, 2\}, M = \{f : X \to \mathbb{Z}\}$
   Note that $h : M \to \mathbb{Z} \times \mathbb{Z}$ defined by $h(f) = (f(1), f(2))$ is a bijection with inverse given by $h^{-1}((m, n))(1) = m, h^{-1}((m, n))(2) = n$ for $(m, n) \in \mathbb{Z} \times \mathbb{Z}$.
   Hence it is enough to show that $\mathbb{Z} \times \mathbb{Z}$ is countable.
   Note that $\mathbb{Z} \times \mathbb{Z} = \cup_{n \in \mathbb{Z}} \{n\} \times \mathbb{Z}$. That is $\mathbb{Z} \times \mathbb{Z}$ is a countable union of countable sets. Hence $\mathbb{Z} \times \mathbb{Z}$ is countable. $\square$

3. *Show that given any real number x there exists a natural number n such that $n > x$*

   Solution: Let $x \in \mathbb{R}$. Assume that $n \leq x$ for all $n \in \mathbb{N}$. Therefore x is an upper bound of $\mathbb{N}$. Therefore by completeness property, $\mathbb{N}$ has a supremum $u \in \mathbb{R}$. Then $u - 1 < u$. Therefore $u - 1$ is not an upper bound of $\mathbb{N}$, so there exists $m \in \mathbb{N}$ such that $u - 1 < m$. This implies that $u < m + 1$. But since $m + 1 \in \mathbb{N}$, this contradicts the fact that u is an upper bound of $\mathbb{N}$. Thus there is a $n \in \mathbb{N}$ such that $n > x$. $\square$

4. *Suppose $u : [0, 1] \to \mathbb{R}$ is a continuous function. Define a function $v : [0, 2] \to \mathbb{R}$ by*

$$v(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq 1; \\ u(2-t) & \text{if } 1 < t \leq 2 \end{cases}$$

   Show that v is continuous.

Solution: Given that $u : [0,1] \to \mathbb{R}$ is a continuous function. Define $v : [0,2] \to \mathbb{R}$ by

$$v(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq 1 \\ u(2-t) & \text{if } 1 < t \leq 2 \end{cases}$$

Since $t \mapsto 2-t$ is continuous, and composition of two continuous functions is continuous, clearly v is continuous on $[0,1) \cup (1,2]$. We shall show that v is continuous at 1.

$$\lim_{t\to 1^-} v(t) = \lim_{t\to 1^-} u(t) = u(1) = 1$$

$$\lim_{t\to 1^+} v(t) = \lim_{t\to 1^+} u(2-t) = \lim_{t\to 1^-} u(t) = u(1) = 1$$

Thus $\lim_{t\to 1} v(t) = 1 = v(1)$. Hence v is continuous at 1. $\square$

5. *Let $\{b_n\}_{n\geq 1}$ and $\{c_n\}_{n\geq 1}$ be two sequences of real numbers. Define a new sequence $\{a_n\}_{n\geq 1}$ by*

$$a_n = \begin{cases} b_n & \textit{if } n \textit{ is odd} \\ c_n & \textit{if } n \textit{ is even} \end{cases}$$

Show that if $\{b_n\}, \{c_n\}$ are convergent with the same limit then $\{a_n\}$ is convergent. However, the converse is not true.

Solution: Let $\lim_{n\to\infty} b_n = a = \lim_{n\to\infty} c_n$. We shall show that $\lim_{n\to\infty} a_n = a$. Let $\epsilon > 0$. There exists $N, M \in \mathbb{N}$ such that

$$|b_n - a| < \epsilon \text{ for all } n \geq N, |c_m - a| < \epsilon \text{ for all } m \geq M$$

Let $K = \max\{N, M\}$. Then

$$|b_n - a| < \epsilon \text{ for all } n \geq K, |c_n - a| < \epsilon \text{ for all } n \geq K$$

Thus

$$|a_n - a| < \epsilon \text{ for all } n \geq 2K$$

Hence $\lim_{n\to\infty} a_n = a$.
The converse is also true: If $\{a_n\}_{n\geq 0}$ converges to a then the sequences $\{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ also converges to a. Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that

$$|a_n - a| < \epsilon \text{ for all } n \geq N$$

This implies that

$$|b_n - a| < \epsilon \text{ for all } n \geq N, |c_m - a| < \epsilon \text{ for all } m \geq N$$

Thus $\lim_{n\to\infty} b_n = a = \lim_{n\to\infty} c_n$. $\square$

6. *Suppose $g : [0,1] \to [3,5]$ is a continuous bijection. Show that either g is strictly increasing with $g(0) = 3, g(1) = 5$ or g is strictly decreasing with $g(0) = 5, g(1) = 3$. Give an example of a discontinuous bijection from $[0,1]$ to $[3,5]$ which is not monotonic.*

Solution: Since g is a bijection it is enough to show that g is either stricly increasing or strictly decreasing.
Suppose this is not the case, without loss of generality, there exists $x, y, z \in [0,1], x < y < z$ such that $g(x) < g(y), g(z) < g(y)$. Since $g(x) \leq \max\{g(x), g(z)\} < g(y), g(z) \leq \max\{g(x), g(z)\} < g(y)$,

and g is continuous, by intermediate value theorem there exist $u \in [x, y), v \in (y, z]$ such that $g(u) = g(v) = \max\{g(x), g(z)\}$. But $u \neq v$. This is a contradiction to g being injective. Hence g is either strictly increasing or strictly decreasing.
Example: Define $f : [0, 1] \to [3, 5]$ by

$$f(t) = \begin{cases} 2t + 3 & \text{if } 0 \leq t < \frac{1}{2}, \\ -2t + 6 & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then f is a bijection from $[0, 1]$ to $[3, 5]$ which is not monotonic.

7. *Find* $\liminf$ *and* $\limsup$ *of the following sequences (you should also prove your claims):*

   (a) $\{x_n\}_{n\geq 1}$, *with* $x_n = (-\frac{1}{2})^n + \frac{1}{n}$ *for* $n \geq 1$.

   (b) $\{y_n\}_{n\geq 1}$, *with* $y_n = (-1)^n \frac{2n^2+3n+1}{5n^2+4}$ *for* $n \geq 1$.

   Solution:

   (a) Given that $x_n = (-\frac{1}{2})^n + \frac{1}{n}$ for $n \geq 1$. Note that $0 \leq x_n \leq \frac{2}{n}$ and $\lim_{n\to\infty} \frac{2}{n} = 0$ implies that $\lim_{n\to\infty} x_n = 0$. Hence
   $$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \lim_{n\to\infty} x_n = 0$$

   (b) Note that $y_n = (-1)^n \frac{2n^2+3n+1}{5n^2+4} = (-1)^n \frac{2+\frac{3}{n}+\frac{1}{n^2}}{5+\frac{4}{n^2}}$ for all $n \geq 1$. Let $z_n = \frac{2+\frac{3}{n}+\frac{1}{n^2}}{5+\frac{4}{n^2}}$. Then $\lim_{n\to\infty} z_n = \frac{2}{5}$ and $y_n = (-1)^n z_n$. Thus $\lim_{n\to\infty} y_{2n+1} = -\frac{2}{5}$ and $\lim_{n\to\infty} y_{2n} = \frac{2}{5}$. Note that
   $$\inf_{k\geq m} \{y_k\} = \inf_{2k+1\geq m} \{y_{2k+1}\} \text{ and } \sup_{k\geq m} \{y_k\} = \sup_{2k\geq m} \{y_{2k}\}$$

   Hence
   $$\liminf_{n\to\infty} y_n = \lim_{m\to\infty} \inf_{k\geq m} \{y_k\} = \lim_{m\to\infty} \inf_{2k+1\geq m} \{y_{2k+1}\} = \liminf_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} y_{2n+1} = -\frac{2}{5}$$
   $$\limsup_{n\to\infty} y_n = \lim_{m\to\infty} \sup_{k\geq m} \{y_k\} = \lim_{m\to\infty} \sup_{2k\geq m} \{y_{2k}\} = \limsup_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} y_{2n} = \frac{2}{5}$$

$\square$